Math 2050A, 2022/23, Solution of Test 1

Answer the following four questions (35%+20%+20%+25%)

1. (a) State the completeness of $\mathbb{R}$;

   Solution: For every **nonempty subset** S of $\mathbb{R}$, if S is **bounded from above**, then **sup S** exists.

   (b) Let $A = \{x \in \mathbb{R} : x^3 + x < 1\}$.

   i. Show that A is non-empty and bounded from above but not bounded from below.

   Solution: Since $0^3 + 0 = 0 < 1$, we know that $0 \in A$. Now note that $\forall x \geq 1$, $x^3 + x \geq 1^3 + 1 = 2$ and $x \notin A$, that is, $\forall x \in A$, $x < 1$, so A is bounded from above. Since $\forall x \leq 0$, $x^3 + x \geq 0^3 + 0 = 0$, and then $x \in A$. If b is a lower bound of A, $b \leq 0$ since $0 \in A$, and now we can take an integer $n > -b \geq 0$ by the Archemedian property of the natural number, then $-n \in A$ and $-n < b$, a contradiction.

   ii. Show that there exists two sequence of $x_n \in A$ and $y_n \notin A$ such that $\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} y_n = \sup A$.

   Solution: By the completeness of $\mathbb{R}$, sup A exists. Let $y_n = \sup A + \dfrac{1}{n} > \sup A$, $y_n \notin A$ and $\lim_{n\to+\infty} y_n = \sup A$. $\forall n \in \mathbb{N}$, $\sup A - \dfrac{1}{n}$ is not an upper bound, so $\exists x_n \in A$ such that $\sup A - \dfrac{1}{n} < x_n < \sup A$, and then $\lim_{n\to+\infty} x_n = \sup A$

   iii. Using ii, show that $\sup A = u$ satisfies $u^3 + u = 1$. (you may use the algebraic properties of convergence.)

   Solution: Since $x_n \in A$, we have $x_n^3 + x_n < 1$, and

$$\lim_{n\to+\infty} x_n^3 + x_n = \left(\lim_{n\to+\infty} x_n\right)^3 + \lim_{n\to+\infty} x_n = u^3 + u \leq 1.$$

   Similarly, $y_n \notin A$, we have $y_n^3 + y_n \geq 1$ and

$$\lim_{n\to+\infty} y_n^3 + y_n = \left(\lim_{n\to+\infty} y_n\right)^3 + \lim_{n\to+\infty} y_n = u^3 + u \geq 1.$$

   Hence we conclude that $u^3 + u = 1$.

2. Using the ε, N terminology, show that

$$\lim_{n\to+\infty} \frac{n^3 + 2n + 1}{4^n} = 0.$$

Solution: By the binomial theorem, when $n \geq 4$, we have that

$$\begin{aligned}
4^n =& (1+3)^n = 1 + 3\binom{n}{1} + 9\binom{n}{2} + 27\binom{n}{3} + 81\binom{n}{4} + \cdots \\
>& 81\binom{n}{4} = \frac{81n(n-1)(n-2)(n-3)}{24} \\
>& n(n-1)(n-2)(n-3)
\end{aligned}$$

When $n \geq 6$, $n(n-1)(n-2)(n-3) \geq n(n-\frac{n}{2})(n-\frac{n}{2})(n-\frac{n}{2}) = \frac{n^4}{8}$, and then we have (when $n \geq 6$)

$$\left|\frac{n^3+2n+1}{4^n}\right| < \frac{8(n^3+2n+1)}{n^4} = \frac{8}{n} + \frac{16}{n^3} + \frac{8}{n^4} < \frac{8}{n} + \frac{1}{n} + \frac{1}{n} = \frac{10}{n}.$$

For $\epsilon > 0$, take $N > \max\{\frac{10}{\epsilon}+1, 6\}$, then $\forall n > N$, $\left|\frac{n^3+2n+1}{4^n}\right| < \frac{10}{n} < \epsilon$. Hence $\lim_{n\to+\infty} \frac{n^3+2n+1}{4^n} = 0$.

3. Prove the following without using monotone convergence theorem: Suppose $\{x_n\}_{n=1}^{\infty}$ is a sequence such that it is bounded from below and $x_{n+1} \leq x_n$ for all $n \geq 1$, then it is convergent.

   Solution: $S = \{x_n\}_{n=1}^{\infty}$ is a nonempty subset of $\mathbb{R}$ that is bounded from below, so $\inf S$ exists by the completeness of $\mathbb{R}$. Denote $a = \inf S$. Now $\forall \epsilon > 0$, $a + \epsilon$ is not a lower bound, so $\exists k$, such that $x_k < a + \epsilon$, and $a \leq x_k$. $\forall n \geq k$, $a \leq x_n \leq x_k < a + epsilon$, so $|x_n - a| = x_n - a < \epsilon$, that is, $\lim_{n\to+\infty} x_n = a$.

4. Define a sequence of real number by $x_1 = 2$ and

$$x_{n+1} = (2x_n)^{1/3}, \quad \forall n \geq 1.$$

   (a) Using monotone convergence theorem, show that $\{x_n\}$ is convergent.

   Solution: We will show that $\{x_n\}_{n=1}^{\infty}$ is monotonic decreasing and $x_n \geq \sqrt{2}$, $\forall n \geq 1$ by mathematical induction.

   $x_1 = 2 > x_2 = (2x_1)^{1/3} = \sqrt[3]{4} > \sqrt{2}$. The base is done. Now suppose we have that $x_{k-1} \geq x_k \geq \sqrt{2}$, we have $x_{k+1} = (2x_k)^{1/3} \geq (2\sqrt{2})^{1/3} = \sqrt{2}$, and $x_{k+1} = (2x_k)^{1/3} \leq (2x_{k-1})^{1/3} = x_k$. Hence $\{x_n\}_{n=1}^{\infty}$ is monotonic decreasing and $x_n \geq \sqrt{2}$, $\forall n \geq 1$.

   Since $\{x_n\}_{n=1}^{\infty}$ is monotonic decreasing and bounded from below by $\sqrt{2}$, $\lim_{n\to+\infty} x_n$ exists.

   (b) Find the limit of the convergent sequence. Justify your answer.

   Solution: Since $\lim_{n\to+\infty} x_n$ exists (denoted by a), we can take the limits of both sides of the equation $x_{n+1}^3 = 2x_n$. By the algebraic properties of limit, we have that

$$\lim x_{n+1}^3 = (\lim x_n)^3 = a^3 = \lim 2x_n = 2\lim x_n = 2a.$$

   The equation has three solutions $0, \sqrt{2}, -\sqrt{2}$. Since $x_n \geq \sqrt{2}$, we have that $a \geq \sqrt{2}$, so $a = \sqrt{2}$.